# On Doubly Periodic Tangles

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# Doubly Periodic Tangles

- Doubly periodic tangles are entanglements of curves embedded in the thickened plane that are periodically repeated in two transversal directions.
- Such entangled networks are useful in many scientific fields for the study of physical systems, and inspire **new mathematical developments**.
- A better understanding of their **geometry and topology**, often associated to some **physical properties**, could allow the prediction of functions during the design process.
- No universal **mathematical study** and many **open questions**.

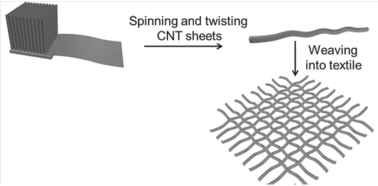



Figure: Carbon Nanotube Woven Fabric (DOI: 10.1007/978-3-319-26893-4)

# Doubly Periodic Tangles - Motivation

DP tangles serve as a significant framework for analyzing and understanding the topological properties of interwoven filament systems across micro-, meso- and macro-scales, including, but not limited to:

- polymer melts

  E. Panagiotou, K.C. Millett, S. Lambropoulou, Quantifying Entanglement for Collections of Chains in Models with Periodic Boundary Conditions,*Procedia IUTAM.*, **7** (2013), 251–260.

- fabric-like structures

  S.G. Markande, E. Matsumoto, Knotty Knits are Tangles in Tori, *Proc. of Bridges* (2020) 103–112.

- molecular chemistry

  Y. Liu, M. O'Keeffe, M.M.J. Treacy and O.M. Yaghi. The geometry of periodic knots, polycatenanes and weaving from a chemical perspective: a library for reticular chemistry. *Chem. Soc. Rev.* **47** (2018), 4642–4664.

- cosmic filaments

  J. R. Bond, L. Kofman and D. Pogosyan, How filaments of galaxies are woven into the cosmic web, *Nature*, **380**, 603-606 (1996).

# Doubly Periodic Tangles and Motifs

## Definition

Let $\tau$ be a link in $T^2 \times I$. A *doubly periodic tangle*, or *DP tangle*, is the lift of $\tau$ under the covering map $\rho$, and is denoted by $\tau_\infty$. Moreover, the projection of $\tau$ onto $T^2 \times \{0\}$ is called a *link diagram* of $\tau$, denoted by $d$, and the lift of $d$ under $\rho$ is called a *doubly periodic diagram*, or *DP diagram*, denoted by $d_\infty$. In this context, $d$ (resp. $\tau$) is called a *motif* for $d_\infty$ (resp. for $\tau_\infty$).

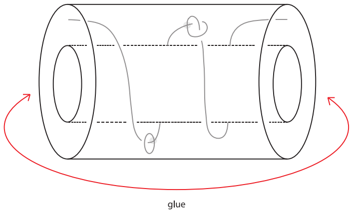



Figure: A link in $T^2 \times I$.

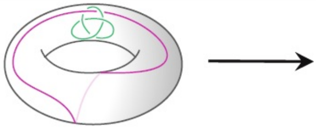
# Doubly Periodic Tangles and Motifs

## Definition

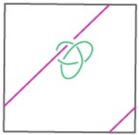
Let $\tau$ be a link in $T^2 \times I$. A *doubly periodic tangle*, or *DP tangle*, is the lift of $\tau$ under the covering map $\rho$, and is denoted by $\tau_\infty$. Moreover, the projection of $\tau$ onto $T^2 \times \{0\}$ is called a *link diagram* of $\tau$, denoted by $d$, and the lift of $d$ under $\rho$ is called a *doubly periodic diagram*, or *DP diagram*, denoted by $d_\infty$. In this context, $d$ (resp. $\tau$) is called a *motif* for $d_\infty$ (resp. for $\tau_\infty$).

## Definition

The representation of a motif $d$ on a flat torus is called *flat motif*.



A link in the thickened torus / a diagram on the torus = a motif



A flat motif

# DP Tangles and Motifs

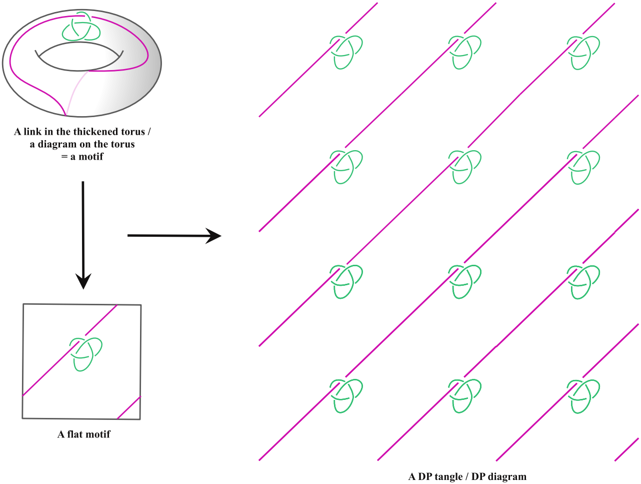
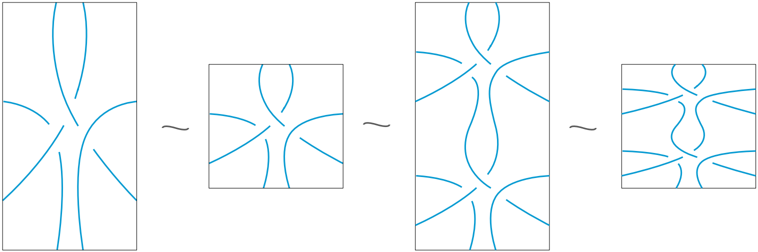



Figure: A link in $T^2 \times I$, a corresponding flat motif and its DP tangle.

- The topological classification of DP tangles is at least as hard a problem as the full classification of knots and links in the three-space.
- Such problems are approached by constructing topological invariants.
- The equivalence relation that these invariants respect is based on assumptions of **minimal motifs**, that is motifs that are minimal for reproducing the DP tangles under 2-periodic boundary conditions.
- Note however that obtaining a minimal motif of a DP tangle is known to be a non trivial problem.

# Different motifs BUT Same DP Tangle

Observe that the same DP tangle can be generated by infinitely many different motifs (associated to all possible finite covers).



(a) (b) (c) (d)

A DP tangle comes equipped with a basis of the plane and a choice of a longitude-meridian pair (l,m) for $T^2$. By a general position argument, m and l do not intersect crossings of d and no arc of d intersects a corner formed by $m$ and $l$.

# Supporting Lattices

Let $B = \{u, v\}$ be a basis of $\mathbb{E}^2$ such that $\rho(u) = l$ and $\rho(v) = m$ for $T^2$. The set of points $\Lambda(u, v) = \{xu + yv \,|\, x, y \in \mathbb{Z}\}$ generated by $B = \{u, v\}$ defines a *periodic lattice* $\Lambda$ for a DP tangle.
Moreover, two bases $B = \{u, v\}$ and $B' = \{u', v'\}$ generate the same point lattice $\Lambda = \Lambda(u, v) = \Lambda'(u', v')$ if and only if for $x_1, x_2, x_3, x_4 \in \mathbb{Z}$,

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} \text{ , \textit{where} } \mid x_1 x_4 - x_2 x_3 \mid = \pm 1.$$

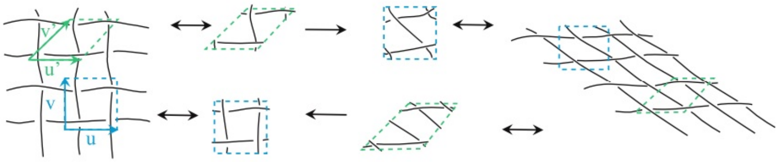



Figure: A DP diagram with a fixed lattice and two different bases of $E^2$ leading to a shearing of the DP tangle

# Supporting Lattices

## Definition

We define the *minimal lattice* of a DP tangle $\tau_\infty$, denoted by $\Lambda_{min}$, to be the lattice satisfying $\Lambda \subseteq \Lambda_{min}$, for all periodic lattices $\Lambda$ of $d_\infty$. Accordingly, the motif (resp. flat motif) $d = d_\infty / \Lambda_{min}$ is called a *minimal(flat) motif for* $d_\infty$.

Hence, a minimal lattice is a minimal 'grid' that accommodates a DP tangle and a minimal motif (resp. flat motif) is minimal for generating the DP diagram d.

# DP equivalence

We now study the equivalence of DP tangles as reflected in their flat motifs.

## Doubly periodic isotopies

- local motif isotopies (that is, surface isotopies and Reidemeister moves).

- global isotopies, induced by invertible affine transformations of the plane.

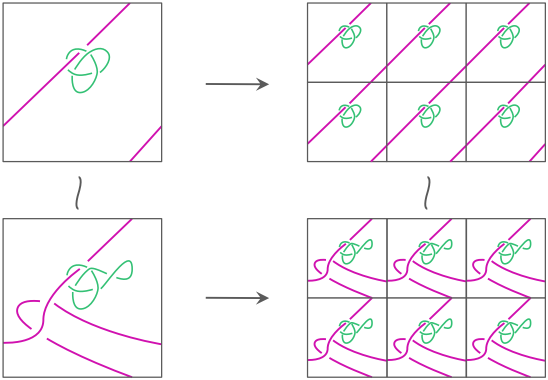
  These consist of both non-area preserving transformations such as re-scalings (stretches, contractions), as well as area preserving transformations like rigid translations and rotations of the plane, or shear deformations.

# Local Motif Isotopies

Let $\tau$ be a motif in a fixed thickened torus $T^2 \times I$ that generates $\tau_\infty$ and $d$ the corresponding diagram of $\tau$ in $T^2$. With $T^2 \times I$ fixed, any isotopy of $\tau$ generates naturally a DP isotopy of $\tau_\infty$. Hence it induces a local isotopy equivalence relation between the corresponding flat motifs, which are supported by the same fixed lattice $\Lambda(u, v)$. On the diagrammatic level, an isotopy of $\tau$ translates into a finite sequence of local moves on the diagram $d$, comprising local surface isotopies and the Reidemeister moves.

# Local Motif Isotopies

These moves on the diagram $d$ generate, in turn, on $d_\infty$ (local) planar isotopies and Reidemeister moves, that preserve the double periodicity, that is, DP (local) planar isotopies and DP Reidemeister moves.
Clearly the above apply to any finite cover of the motif.



# Surface Isotopies

We now examine local surface isotopies. These on the flat motif level comprise:

- a) planar isotopies within the flat motif,
- b) planar isotopies where an arc before lies within the motif but the arc afterwards hits one boundary component.

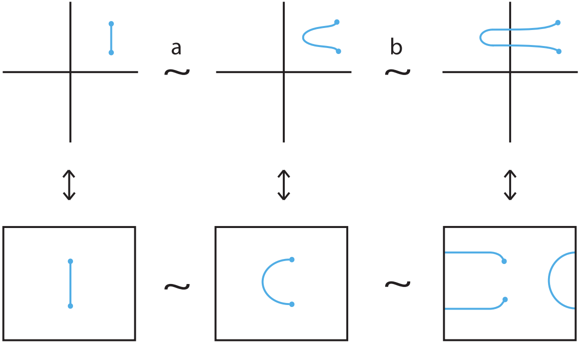
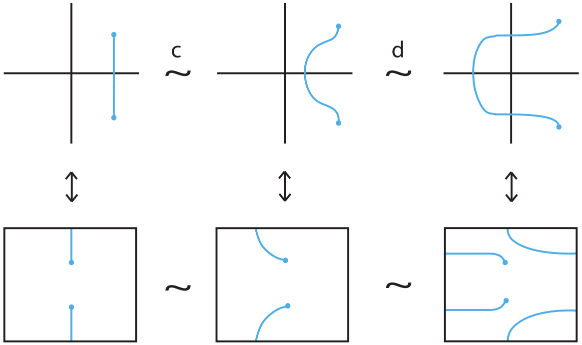



Figure: Surface isotopies of types a) and b).

# Surface Isotopies

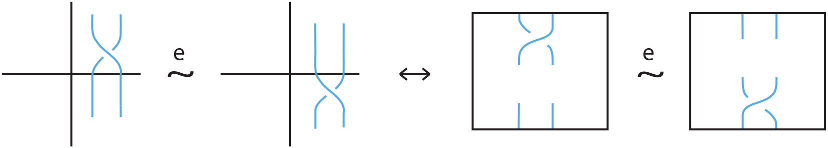
We now examine local surface isotopies. These on the flat motif level comprise:

- c) planar isotopies where an arc before and the arc afterwards cross one boundary component,
- d) planar isotopies where an arc before crosses one boundary component but the arc afterwards crosses both boundary components.



# Surface Isotopies

- e) the situation where a crossing passes through one boundary component:



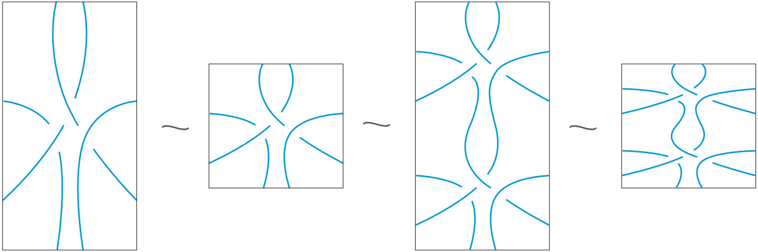

**Remark**

The local isotopy moves between DP tangles that are discussed so far, correspond to local moves between motifs on the fixed torus $T^2$, thus between flat motifs supported by the same lattice.

# Torus Re-Scaling

DP stretches and contractions of the DP tangle $\tau_\infty$ are also DP isotopies. On the level of motifs, they are induced by analogous re-scaling isotopies of the supporting torus.

Re-scalings correspond to torus isotopies, like inflation or contractions (blow-ups and shrinkings):



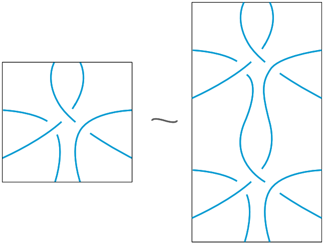
(a) (b) (c) (d)

# Scale Equivalence

It is straightforward that any finite cover of a given flat motif is also a flat motif for the same DP tangle.

## Definition

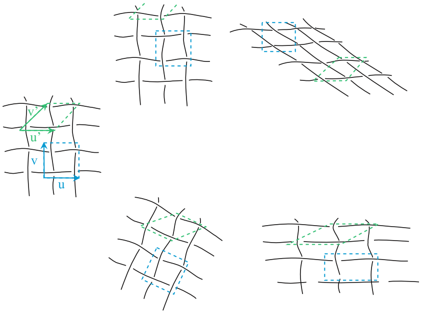
Let $d_\infty$ be a DP diagram and let $\Lambda_0$, $\Lambda_1$ and $\Lambda_2$ be three (not necessarily distinct) point lattices such that $\Lambda_1 \subseteq \Lambda_0$ and $\Lambda_2 \subseteq \Lambda_0$. Moreover, let $d_0 = d_\infty/\Lambda_0$, $d_1 = d_\infty/\Lambda_1$ and $d_2 = d_\infty/\Lambda_2$ be flat motifs of $d_\infty$. Then, $d_1$ and $d_2$ arise as *adjacent* copies of $d_0$, according to the inclusion relations of the lattices, and $d_1$ and $d_2$ are said to be *scale equivalent*.



The above definition leads to the question of **motif minimality**.

# Affine Transformations of the Plane

- Re-scaling transformation
- Rigid translation
- Rigid rotation
- Shearing transformation

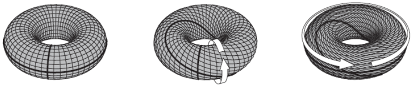


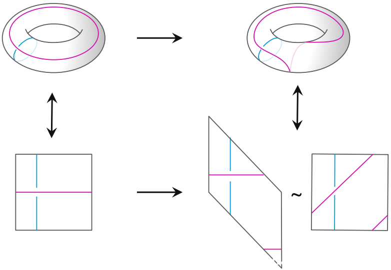

Any affine transformation of the plane is generated by the above elementary transformations.

# Dehn Equivalence

Shear deformations correspond to *Dehn twists* of the underlying torus, which are orientation preserving self-homeomorphisms of the torus.

Recall that the point lattice $\Lambda$, with basis $B = \{u\ ,\ v\}$, can be generated by different bases of $\mathbb{E}^2$, associated to different flat motifs for the same DP tangle. Let $B' = \{u'\ ,\ v'\}$ be a new basis of $\mathbb{E}^2$, inducing a shearing of the plane. Assign the longitude $l$ of $T^2$ to $u'$ and the meridian $m$ to $v'$. This creates a new motif $d'$, which differs from the motif $d$ associated to the basis $B$ by a finite sequence of *Dehn twists*.





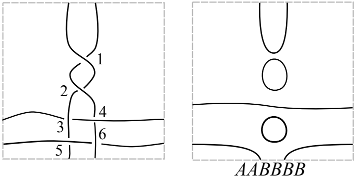
# Main Theorem for DP Tangle Equivalence

**Theorem (ID, Lambropoulou and Mahmoudi)**

***(DP Tangle Equivalence)***: *Let $\tau_{1,\infty}$ and $\tau_{2,\infty}$ be two DP tangles in $\mathbb{E}^2 \times I$, with corresponding DP diagrams $d_{1,\infty}$ and $d_{2,\infty}$. Let also $\Lambda_1$ and $\Lambda_2$ be the supporting point lattices such that $d_i = d_{i,\infty}/\Lambda_i$ is a flat motif of $d_{i,\infty}$ for $i \in \{1, 2\}$. Then $\tau_{1,\infty}$ and $\tau_{2,\infty}$ are equivalent if and only if $d_1$ and $d_2$ are related by a finite sequence of* **shifts, motif isotopy moves, Dehn twists, orientation preserving affine transformations and scale equivalence**.

# DP Bracket Polynomial

[Grishanov, Meshkov, Omelchenko, 2007]


$$
\begin{aligned}
f(\tau) &= (-A)^{-3w_r(\tau)}\langle \tau \rangle, \\
&= (-A)^{-3w_r(\tau)}\left(\sum_S A^{i-j}(-A^{2}\smile A^{-2})^{c_S}(m,n)_S\right). \qquad (1)
\end{aligned}
$$

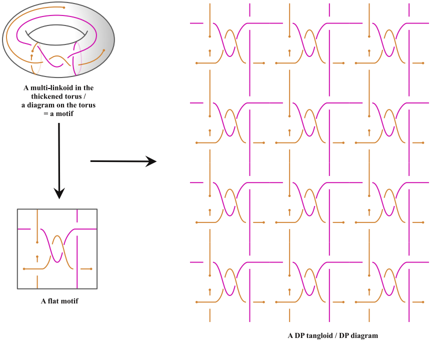
## Theorem

*The polynomial $f(\tau) \in \mathbb{Z}[A]$ defined above is an invariant for oriented DP tangles under motif isotopies for a fixed point lattice.*

# Other Diagrammatic Settings

We have also generalized these results to other diagrammatic settings. In particular:

- regular and framed isotopies,
- virtual DP tangles,
- singular & pseudo DP tangles,
- tied & bonded DP tangles.
- DP tangloids (linkoids).

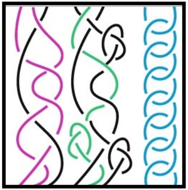


# Directional Invariants

## Motivation

Each component of a link embedded in the torus surface $T^2$ is a closed curve, which is either *null-homotopic*, namely contractible to a point in $T^2$, or *essential*, namely non-contractible in $T^2$.

The null-homotopic components can be isotoped to circles, while the essential components comprise torus knots and torus links.



# Interlinked Compounds

## Definition

Let $\tau$ be a (flat) motif in $T^2 \times I$ and $\mu$ be a set of closed components of $\tau$ such that,

i. $\mu$ forms a split sublink of $\tau$, that is, a sublink that can split from the rest of $\tau$ by isotopy;

ii. $\mu$ cannot split into separate sublinks under isotopy.

Then, $\mu$ is said to be an *interlinked compound* of $\tau$.

Figure: Examples of flat motifs with 1 interlinked compound.

Figure: A flat motif with 2 interlinked compounds.

Figure: Examples of flat motifs with 3 interlinked compounds.

## Definition

An interlinked compound $\mu$ of a (flat) motif $\tau$ or a DP tangle $\tau_\infty$ belongs to one of the following classes:

a. $\mu$ can be enclosed, up to motif isotopy, in a 3-ball in $T^2 \times I$. Then it is referred to as *null-homotopic compound* of $\tau$ and it lifts in $\mathbb{E}^2 \times I$ to a disjoint union of null-homotopic compounds.

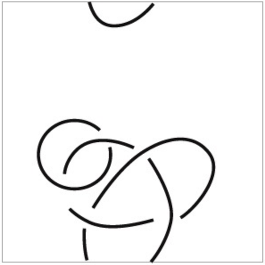


Figure: A null-homotopic compound.

## Definition

An interlinked compound $\mu$ of a (flat) motif $\tau$ or a DP tangle $\tau_\infty$ belongs to one of the following classes:

b. $\mu$ can be enclosed, up to motif isotopy, in an essential thickened ribbon in $T^2 \times I$. In this case $\mu$ is referred to as *ribbon compound* of $\tau$ and it lifts in $\mathbb{E}^2 \times I$ to an infinite disjoint union of identical ribbon compounds in $\tau_\infty$.

Figure: A ribbon compound.

## Definition

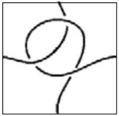
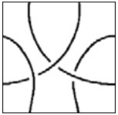
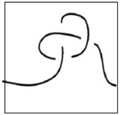
An interlinked compound $\mu$ of a (flat) motif $\tau$ or a DP tangle $\tau_\infty$ belongs to one of the following classes:

c. $\mu$ cannot be enclosed in either a 3-ball or a thickened ribbon in $T^2 \times I$, under any motif isotopy. In this case $\mu$ is referred to as *cover compound* of $\tau$ and it lifts in $\mathbb{E}^2 \times I$ to a single cover compound in $\tau_\infty$.

Figure: Examples of cover compounds.

## Proposition

The class of an interlinked compound as null-homotopic, ribbon or cover is an invariant under motif equivalence resp. DP tangle equivalence.

| | **Minimal motif (lattice)** | **Crossing number** | **Number of components** | **Class (subclasses)** |
|---|---|---|---|---|
| 1 |  | 3 | 1 | Cover (essential) |
| 2 |  | 3 | 1 | Cover (essential) |
| 3 |  | 3 | 1 | Ribbon (essential) |

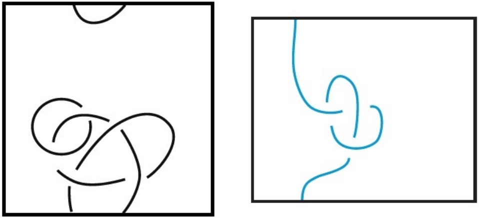
We now introduce different types of DP tangles based on the directions of elementary constituents of theirs.

## Definition

The *direction of an element* is defined as follows. Let $e$ be an element of a (flat) motif $\tau$. Then:

a. if $e$ is an *isolated knot*, thus homotopic to the trivial knot, then $e$ is said to have direction $(0, 0)$.

b. if $e$ is an *essential component*, homotopic to an $(a, b)$-torus knot, then $e$ is said to have direction $(a, b)$.



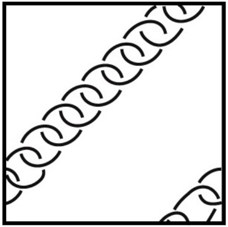
# Directions

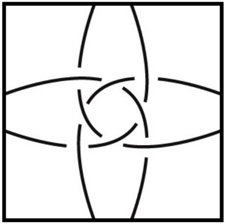
## Definition

The *direction of an element* is defined as follows. Let $e$ be an element of a (flat) motif $\tau$. Then:

c. if $e$ is a *chain-link*, whose centerline curve is isotopic to an $(a,b)$-torus knot, then $e$ is said to have direction $(a,b)$.

d. if $e$ is a *full polycatenane*, then $e$ is said to have direction $(\infty,\infty)$.





## Definition

Let $\tau$ be a (flat) motif of a DP tangle $\tau_\infty$. We define the direction of $\tau$ to be the set of directions of its elements, taken up to homotopy in the flat motif. Furthermore, the direction of $\tau_\infty$ is defined to be the same as the direction of $\tau$.

## Remark

The direction of a (flat) motif (resp. a DP tangle) encodes the class of the motif (resp. the DP tangle).

A geometrical representation of the direction of a motif, that we call an axis-motif, is obtained by replacing elements of the motif by null-homotopic closed curves or essential closed curves.

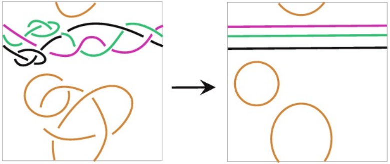
## Definition

The *axis of an interlinked compound* is defined as follows. Let $\mu$ be an interlinked compound of a (flat) motif $\tau$. Then:

a. if $\mu$ is a null-homotopic compound of $\tau$ defined as a link of $c$ knots, then the axis of $\mu$ is a trivial link of $c$ components.

b. if $\mu$ is a ribbon compound of $\tau$ defined as a link of $k$ elements with direction $(a, b)$ and $c$ isolated knots, where $c = 0, 1, \ldots$, then the axis of $\mu$ consists of a $(ka, kb)$-torus link and of a trivial link of $c$ components.

c. if $\mu$ is a cover compound of $\tau$, then the axis of $\mu$ is the set of the axes of all its elements.



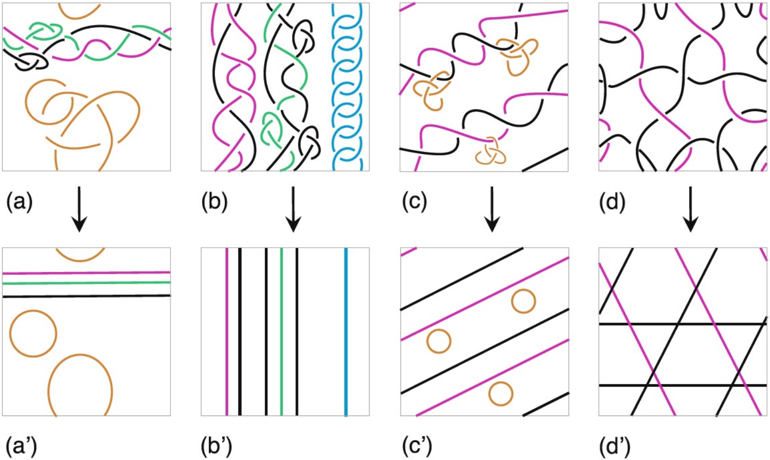
## Definition

The *axis-motif* $\alpha(\tau)$ *of a (flat) motif* $\tau$ is defined as the projection of a (flat) motif formed by replacing each element of $\tau$ by its axis, and the axes are taken up to homotopy preserving the orders of the intersections with longitude-meridian. An axis motif that corresponds to a minimal motif shall be called *minimal axis-motif of* $\tau$.

## Theorem

*The total number of distinct directions of a (flat) motif resp. DP tangle is a topological invariant of the DP tangle.*

# Examples



THANK YOU FOR READING!

## References


- I. D., S. Lambropoulou, S. Mahmoudi, "Equivalence of Doubly Periodic Tangles", arXiv: 2310.00822.
- I. D., S. Lambropoulou, S. Mahmoudi, "Directional Invariants of Doubly Periodic Structures", Symmetry 2024, 16(8), 968; https://doi.org/10.3390/sym16080968.
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